Kittel effect

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The Kittel effect describes ferromagnetic resonance (FMR) under conditions where the bias magnetic field varies in response to the motion of the magnetization during one precession cycle. A common case occurs in ferromagnetic thin films where the equilibrium magnetization lies in the film plane. In this configuration, both the demagnetizing field and the magnetic field of spin–orbit interaction are modulated during each precession period. These fields reach their maximum values when the magnetization is tilted out of the plane and vanish when it lies entirely in-plane. The Kittel effect modifies the Larmor frequency and gives rise to higher-order harmonics of the FMR signal, particularly the second and third harmonics.. Main part finished in March 2025

Several parameters of ferromagnetic resonance (FMR) are influenced by the Kittel effect. The most well-known manifestation is the non-linear dependence of the Larmor frequency on an external magnetic field applied along the easy axis (i.e., in the in-plane direction). In general, the Larmor frequency is linearly proportional to the total magnetic field, which is the sum of the internal and external magnetic fields. Thus, increasing the external magnetic field normally leads to a linear increase of Larmor frequency. However, the Kittel effect causes this increase to become faster and nonlinear.

The Kittel effect describes how the demagnetization field and the effective field from spin-orbit interaction influence spin precession, leading to an increase in the precession frequency. When the external magnetic field becomes sufficiently large compared to these internal fields, the influence of the Kittel effect becomes negligible and eventually disappears.

Another important consequence of the Kittel effect is the generation of second and third harmonics in the FMR response. This occurs because the Kittel effect introduces additional complexity into the magnetization motion, producing a nonlinear variation of the precession angle, making its temporal evolution more complex.

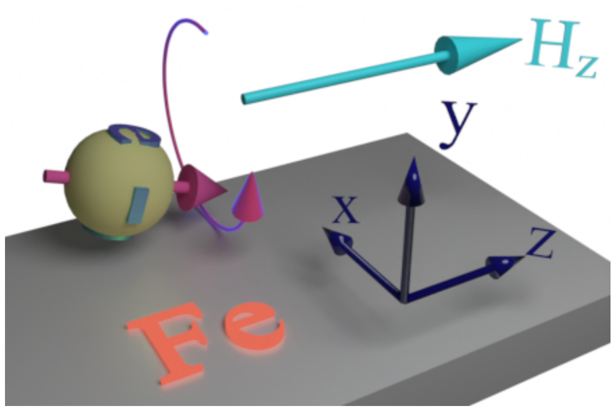

FIG. 1. Precession geometry. The equilibrium magnetization is perpendicular to the film (the z- direction). The external magnetic field H_z is applied along the easy axis (the z- direction).

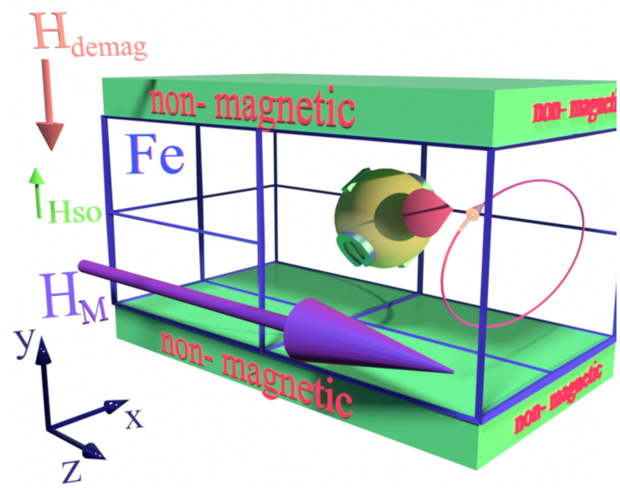

FIG. 2. Precession geometry. The equilibrium magnetization is in-plane (the z- direction). The external magnetic field H_z is applied along the easy axis (the z- direction). The spin creates three types of the magnetic fields: (field 1: H_M): the magnetic field along spin direction (field 2: H_{demag}): Demagnetization field directed along film normal (the y-axis); (field 3: H_{so}) Magnetic field of spin orbit interaction directed along film normal (the y-axis) and opposite to H_{demag}. All magnetic fields are modulated during precession H_{demag} and H_{so} become largest when the magnetization is directed toward the film surface and vanish when magnetization is fully in-plane.

The Kittel effect can be modeled either by solving the Landau-Lifshitz equation or via a quantum-mechanical approach. Section 1 discusses the solution of the Landau-Lifshitz equation as originally formulated by Kittel. This model, while substantially simplified, provides a good approximation for calculating the Larmor frequency. Section 2 presents a full solution of the Landau–Lifshitz equations without approximations, which not only captures the modification of the Larmor frequency but also explains the emergence of the second and third harmonics.

I. GEOMETRY OF THE EFFECT, FINAL RESULTS, LANDAU- LIFSHITZ EQUATION FOR THE EFFECT

A. conditions, assumptions, axis directions

(condition): The equilibrium magnetization is in-plane. It is the case of a relatively thick ferromagnetic field

(axes): The equilibrium magnetization is in the z-direction (the in-plane direction). The interface normal is in the y- direction (See Fig. 1).

B. Use of Two Unit Systems (SI and Gaussian)

To compare the obtained results with the classical Kittel formula, the Gaussian unit system is used. In all other cases, the SI unit system is employed, as it is more convenient for incorporating the effects of spin–orbit interaction and interfacial imperfections.

For the Kittel effect, the only difference between the two unit systems lies in the definition of the dipolar magnetic field produced by the magnetization.

In the SI system, the dipolar magnetic field H_M is calculated as: $H_M = M$, while in the Gaussian system, it is calculated as $H_M = 4\pi M$.

C. classical description: results

The Kittel effect describes the influence of the demagnetization field and the field of the spin-orbit interaction on the Larmor precession of the magnetization.

Kittel effect leads to several modifications of parameters of the ferromagnetic resonance. The first modification is the change of the Larmor frequency ω_L, which is described by the Kittel formula (see Eqs. 26 and 47) as (Gaussian unit system):

$$\omega_L = \gamma \cdot \sqrt{H(H + 4\pi M)} \tag{1}$$

where M is the magnetization, γ is the electron gyromagnetic ratio, $H = H_{ext} + H_{int}$ is the bias in-plane magnetic field, along which the magnetization precession occurs. H is the sum of the internal magnetic field H_{int} and external in-plane magnetic field H_{ext}.

The Kittel formula is valid only in case of the ideally smooth interfaces of the ferromagnetic field and absence of the influence of the spin- orbit interaction. For a realistic surface having some roughness and experiencing some spin- orbit interaction, the Larmor frequency is calculated (See Eq. 81) as (SI unit system):

$$\omega_L = \frac{\gamma}{1 + k_{so}} \cdot \sqrt{H(H + H^0_{ani}\frac{(1 + k_{so})}{2})} \tag{2}$$

where H^0_{ani} is the anisotropy field in absence of an external magnetic field and k_{so} is the coefficient of the spin-orbit interaction.

In absence of spin-orbit interaction or when the spin-orbit interaction is isotropic ($k_{so} = 0$), the Larmor frequency is calculated (See Eq. 83) as (SI unit system):

$$\omega_L = \gamma \cdot \sqrt{H(H + M \cdot k_{demag})} \tag{3}$$

where k_{demag} is the demagnetization coefficient. k_{demag} equals to 1 for an ideally smooth interface of the ferromagnetic film and becomes smaller for a rougher interface.

In the case of an ideally smooth interface of the ferromagnetic film (k_{demag}=1), the expression for the Larmor frequency simplifies to the classical Kittel formula (SI unit system):

$$\omega_L = \gamma \cdot \sqrt{H(H + M)} \tag{4}$$

The second modification due to the Kittel effect is the change of precession path from circular to elliptical. When the precession angle becomes smaller when the magnetization is directed towards the film interface than when it directed fully in the in-plane direction.

Ellipticity of precession trajectory can be calculated as (See Eqs. 33 and 49) :

$$R_{yx} = \sqrt{\frac{H}{H + M}} \tag{5}$$

The ellipticity is defined as

$$R_{yx} = i \cdot \frac{M_y}{M_x} \tag{6}$$

The third modification due to the Kittel effect is appearance of the 2nd and 3rd harmonics of the precession.

The precession dynamic, which includes the 2nd and 3rd harmonics, is described as

$$\begin{aligned} M_x &= M_{x,\omega 1} \cdot e^{i\cdot\omega_L t} + M_{x,\omega 3} \cdot e^{i\cdot 3\omega_L t} \\ M_y &= M_{y,\omega 1} \cdot e^{i\cdot\omega_L t} + M_{y,\omega 3} \cdot e^{i\cdot 3\omega_L t} \\ M_z &= M_{z0} + M_{z,\omega 2} \cdot e^{i\cdot 2\omega_L t} \end{aligned} \tag{7}$$

where similarly the Larmor frequency is calculated as (Gaussian unit system)

$$\omega_L = \gamma \cdot \sqrt{H(H + 4\pi M)} \tag{8}$$

Due to the modulation of the precession angle, the magnetization component along the easy axis M_z is modulated at the second harmonic. The ratio of the magnitude of the second harmonic $M_{z,\omega 2}$ to the magnitude of the first harmonic $M_{x,\omega 1}$ is calculated (See Eq. 88) as follows (SI unit system)

$$M_{z,\omega 2} = -i\frac{M_{x,\omega 1}^2}{2M}\frac{1}{1+\dfrac{H}{H_{ani}^0}\dfrac{2}{1+k_{so}}} \quad (9)$$

In absence of the interfacial spin- orbit interaction $k_{so} = 0$ and an ideally smooth interface (k_{demag}=1), the ratio between components osculating with single and double Larmor frequencies is calculated (See Eq. 51) (Gaussian unit system) as

$$M_{z,\omega 2} = -i\frac{2\pi}{H+4\pi M}M_{x,\omega 1}^2 \quad (10)$$

In turn, this modulation of M_z generates precession components at the third-harmonic frequency. The relationship between the amplitudes of the third harmonic $M_{x,\omega 3}$ and first harmonic $M_{x,\omega 1}$ is given by the following expression (See Eq. 90) (See Eq. 88) (SI unit system) as

$$M_{x,\omega 3} = -i\frac{M_{x,\omega 1}^3}{48\cdot M^2}\frac{1}{\left[1+\dfrac{H}{H_{ani}^0}\dfrac{2}{(1+k_{so})}\right]^2} \quad (11)$$

In absence of the interfacial spin- orbit interaction $k_{so} = 0$ and an ideally smooth interface (k_{demag}=1), the ratio between components osculating with single and triple Larmor frequencies is calculated (See Eqs. 59) (Gaussian unit system) as:

$$M_{x,\omega 3} = -i\tfrac{\pi^2}{3}\tfrac{M_{x,\omega 1}^3}{[H+4\pi M]]^2} \quad (12)$$

Similarly, the precession trajectory at the third harmonic is elliptical. The relation between ellipticity of precession trajectory between 1st harmonic $R_{yx,\omega 1}$ and 3rd harmonic $R_{yx,\omega 3}$ is (See Eq. 53):

$$R_{yx,\omega 3} = i\cdot\frac{M_{y,\omega 3}}{M_{x,\omega 3}} = \frac{R_{yx,\omega 1}}{3} \quad (13)$$

It is important to emphasize that the third harmonic component increases rapidly with the growth of the precession angle. Specifically, the third harmonic is proportional to the cube of the zero harmonic component.

D. Landau- Lifshitz equation for Kittel effect

(condition 1 for classical description): The ideal interface, which creates the demagnetization field opposite and exactly equal to the magnetic field of the magnetization. In the unit, which used in paper:$H_{demag,y} = 4\pi M_y$

$$H_{demag,y} = -4\pi M_y \quad (14)$$

(condition 2 for classical description): there is no magnetic field of spin-orbit interaction

The Landau-Lifshitz (LL) equation describes the dynamic of the magnetization precession, induced by total magnetic field H, which includes the internal and external magnetic fields, and the demagnetization field:

$$\frac{\partial\vec{M}}{\partial t} = -\gamma\vec{M}\times(\vec{H}+\vec{H}_{demag}) \quad (15)$$

where γ is the electron gyromagnetic ratio.

Since the direction of the demagnetization field is along the surface normal and there is no in-plane component of the demagnetization field

$$H_{\vec{demag}} = \begin{pmatrix} 0 \\ H_{demag,y} \\ 0 \end{pmatrix} \quad (16)$$

and the total magnetic field is align in-plane and along the z-axis:

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ H_z \end{pmatrix} \quad (17)$$

Eq. 15 is simplified as

$$\begin{array}{l} \frac{\partial M_x}{\partial t} = -\gamma\left[M_y\cdot H_z - M_z\cdot H_{demag,y}\right] \\ \frac{\partial M_y}{\partial t} = -\gamma\left[-M_x\cdot H_z\right] \\ \frac{\partial M_z}{\partial t} = -\gamma\left[M_x\cdot H_{demag,y}\right] \end{array} \quad (18)$$

Substitution of Eq. 14 for the demagnetization into Eq. 18 gives

$$\begin{array}{l} \frac{\partial M_x}{\partial t} = -\gamma\left[M_y\cdot H_z + M_z\cdot 4\pi M_y\right] \\ \frac{\partial M_y}{\partial t} = -\gamma\left[-M_x\cdot H_z\right] \\ \frac{\partial M_z}{\partial t} = -\gamma\left[-M_x\cdot 4\pi M_y\right] \end{array} \quad (19)$$

II. OVERSIMPLIFIED CLASSICAL DESCRIPTION BY KITTEL

A. Kittel's approximate solution

To simplify the solution of the Landau–Lifshitz equation, Kittel introduced a rather rough approximation. Remarkably, despite its simplicity, this approximation produces results that are nearly identical to the exact solution obtained without any approximations.

(**Kittel's approximation**) : The oversimplified Kittel's approximation states that there is no modulation of magnetization component along the easy axis

$$\frac{\partial M_z}{\partial t} = 0 \quad (20)$$

The Kittel's approximation literally means that the precession angle does not vary in time, which fully contradicts with the main feature of the Kittel's effect.

However, the solution of the Landau- Lifshitz equation becomes much simpler when Kittel's approximation is used. Differentiation of 1st Eq of Eqs. 19 and taking into account that the external field is fixed $\frac{\partial H_z}{\partial t} = 0$ and the Kittel approximation (Eq.20) give

$$\frac{\partial^2 M_x}{\partial t^2} = -\gamma \left[H_z \frac{\partial M_y}{\partial t} + 4\pi M_z \frac{\partial M_y}{\partial t} \right] \tag{21}$$

or

$$\frac{\partial^2 M_x}{\partial t^2} = -\gamma \left[H_z + 4\pi M_z \right] \frac{\partial M_y}{\partial t} \tag{22}$$

Substitution of the 2nd Eq of Eqs. 19 into Eq. 22 gives

$$\frac{\partial^2 M_x}{\partial t^2} = -\gamma^2 \left[H_z + 4\pi M_z \right] \cdot M_x \cdot H_z \tag{23}$$

The Eq. 24 can written as

$$\frac{\partial^2 M_x}{\partial t^2} = -\omega_L^2 \cdot M_x \tag{24}$$

where ω_L is Larmor frequency calculated as

$$\omega_L = \gamma \cdot \sqrt{H_z (H_z + 4\pi M_z)} \tag{25}$$

Since the magnetic field is applied only along the easy axis $H = H_z$ and the precession angle is small $M_z \approx M$, Eq. 25 becomes

$$\omega_L = \gamma \cdot \sqrt{H \cdot (H + 4\pi M)} \tag{26}$$

Eq. 26 is the final Kittel formula for the Larmor frequency

It is important to note that the Kittel effect leads to an increase of the Larmor frequency.

B. Oscillation of the precession angle within the Kittel's approximate solution

The solution of Eq. 24 gives the in-plane component of the magnetization M_x, which perpendicular to the easy axis (the z-axis) and which oscillates with the Larmor frequency ω_L, as

$$M_x = M_{x0} \cdot \cos[\omega_L \cdot t + \phi] \tag{27}$$

where ϕ is the precession phase and M_{x0} is the oscillation amplitude

The oscillation of another perpendicular- to- easy- axis components M_y, which is perpendicular to the film plane, is smaller, because of the effect of the demagnetization field.

This component can be calculated from Eqs. 19. Substitution of Eq. 27 into 1st Eq. of Eqs. 19 gives

$$-M_{x0} \cdot \omega_L \cdot \sin[\omega_L \cdot t + \phi] = -\gamma M_y \left[H_z + 4\pi M_z \right] \tag{28}$$

or

$$M_y = \frac{M_{x0} \cdot \omega_L}{\gamma \left[H_z + 4\pi M_z \right]} \sin[\omega_L \cdot t + \phi] \tag{29}$$

Taking into consideration Eq. 26, Eq 29 is simplified as

$$M_y = M_{x0} \frac{\sqrt{H_z (H_z + 4\pi M_z)}}{H_z + 4\pi M_z} \sin[\omega_L \cdot t + \phi] \tag{30}$$

or

$$M_y = M_{x0} \sqrt{\frac{H_z}{H_z + 4\pi M_z}} \sin[\omega_L \cdot t + \phi] \tag{31}$$

or

$$M_y = M_{x0} R_{yx} \sin[\omega_L \cdot t + \phi] \tag{32}$$

where parameter $R_{yx} = \frac{M_{y0}}{M_{x0}}$ is the ration of oscillation amplitudes along perpendicular-to-plane to in-plane directions and which is calculated as

$$R_{yx} = \sqrt{\frac{H_z}{H_z + 4\pi M_z}} \tag{33}$$

Parameter $R_{yx} < 1$, which is always less than 1, shows the decrease of the precession angle when the magnetization is in -direction of the interface normal

The total magnetization component perpendicular to the easy axis $M_\perp$is calculated from Eqs. 27, 31 as

$$\begin{aligned} M_\perp &= \sqrt{M_x^2 + M_y^2} = \\ &= M_{x0} \sqrt{cos^2[\omega_L \cdot t + \phi] + \tfrac{H_z}{H_z + 4\pi M_z} \sin^2[\omega_L \cdot t + \phi]} = \\ &= M_{x0} \sqrt{1 - \tfrac{4\pi M_z}{H_z + 4\pi M_z} \sin^2[\omega_L \cdot t + \phi]} \end{aligned} \tag{34}$$

The magnetization components M_x and M_y define the maximum and minimum precession angles. The average precision angle is established by a balance of the pumping and damping torques.

The magnetization component along the easy axis M_z is calculated as

$$M_\perp = \sqrt{M^2 - M_\perp^2} \tag{35}$$

Therefore, M_z is modulated during the magnetization precession, which is in contradiction with the Kittel Approximation. Taking into account the modulation of M_z leads to generation of 2nd and 3rd harmonics of ω_L, which are calculated in next section by solving LL equations without usage of the Kittel approximation.

III. 2ND AND 3RD HARMONICS OF PRECESSION. SOLUTION OF LANDAU-LIFSHITZ EQUATION FOR KITTEL EFFECT WITHOUT USAGE OF THE KITTEL APPROXIMATION

A solution of the Landau–Lifshitz equation for the Kittel effect does not require the use of Kittel's original rough approximation when higher-order harmonics are included explicitly in the description of the precession. The exact solution of the LL equations for the Kittel effect (Eq. 19) can be expressed as a linear combination of the 1st, 2nd, and 3rd harmonics, as follows:

$$\begin{aligned} M_x &= M_{x,\omega 1} \cdot e^{i \cdot \omega_L t} + M_{x,\omega 3} \cdot e^{i \cdot 3\omega_L t} \\ M_y &= M_{y,\omega 1} \cdot e^{i \cdot \omega_L t} + M_{y,\omega 3} \cdot e^{i \cdot 3\omega_L t} \\ M_z &= M_{z0} + M_{z,\omega 2} \cdot e^{i \cdot 2\omega_L t} \end{aligned} \tag{36}$$

where M_x and M_y components have 1st and 3rd harmonics, while M_z component has zero and 2nd harmonics.

Substitution of Eq. 36 into 1st Eq. of Eqs 19 :

$$\frac{\partial M_x}{\partial t} = -\gamma \cdot [H_z + 4\pi \cdot M_z] \cdot M_y \tag{37}$$

gives

$$\begin{aligned} & M_{x,\omega 1} \cdot i \cdot \omega_L \cdot e^{i \cdot \omega_L t} + M_{x,\omega 3} \cdot i \cdot 3\omega_L \cdot e^{i \cdot 3\omega_L t} = \\ & = -\gamma \left[H_z + 4\pi M_{z0} + 4\pi M_{z,\omega 2} \cdot e^{i \cdot 2\omega_L t} \right] \cdot \\ & \cdot \left[M_{y,\omega 1} \cdot e^{i \cdot \omega_L t} + M_{y,\omega 3} \cdot e^{i \cdot 3\omega_L t} \right] \end{aligned} \tag{38}$$

Comparison of coefficients in Eq. 38 oscillating as $e^{i \cdot \omega_L t}$ and as $e^{i \cdot 3\omega_L t}$ gives

$$\begin{aligned} & i \cdot \omega_L \cdot M_{x,\omega 1} = -\gamma \cdot M_{y,\omega 1} \left[H_z + 4\pi M_{z0} \right] \\ & i \cdot 3\omega_L \cdot M_{x,\omega 3} = -\gamma \cdot M_{y,\omega 3} \left[H_z + 4\pi M_{z0} \right] - \\ & -\gamma \cdot M_{y,\omega 1} \cdot 4\pi M_{z,\omega 2} \end{aligned} \tag{39}$$

Substitution of Eq. 36 into 2nd Eq. of Eqs 19

$$\frac{\partial M_y}{\partial t} = -\gamma \left[-M_x \cdot H_z \right] \tag{40}$$

gives

$$\begin{aligned} & M_{y,\omega 1} \cdot i \cdot \omega_L \cdot e^{i \cdot \omega_L t} + M_{y,\omega 3} \cdot i \cdot 3\omega_L \cdot e^{i \cdot 3\omega_L t} = \\ & = \gamma \cdot H_z \cdot \left[M_{x,\omega 1} \cdot e^{i \cdot \omega_L t} + M_{x,\omega 3} \cdot e^{i \cdot 3\omega_L t} \right] \end{aligned} \tag{41}$$

Comparison of coefficients in Eq. 41 oscillating as $e^{i \cdot \omega_L t}$ and as $e^{i \cdot 3\omega_L t}$ gives

$$\begin{aligned} & i \cdot \omega_L \cdot M_{y,\omega 1} = \gamma \cdot H_z \cdot M_{x,\omega 1} \\ & i \cdot 3\omega_L \cdot M_{y,\omega 3} = \gamma \cdot M_{x,\omega 3} \cdot H_z \end{aligned} \tag{42}$$

Substitution of Eq. 36 into 3nd Eq. of Eqs 19

$$\frac{\partial M_z}{\partial t} = -\gamma \left[-M_x \cdot 4\pi M_y \right] \tag{43}$$

gives

$$M_{z,\omega 2} \cdot i \cdot 2\omega_L \cdot e^{i \cdot 2\omega_L t} = \gamma \cdot 4\pi \cdot M_{x,\omega 1} \cdot e^{i \cdot \omega_L t} M_{y,\omega 1} \cdot e^{i \cdot \omega_L t} \tag{44}$$

Comparison of coefficients in Eq. 44 oscillating as $e^{i \cdot 2\omega_L t}$ gives

$$i \cdot 2\omega_L \cdot M_{z,\omega 2} = 4\pi \cdot \gamma \cdot M_{x,\omega 1} \cdot M_{y,\omega 1} \tag{45}$$

The 1st equation of Eqs 42 and 1st equation of Eqs Eqs.39 makes a system of two equations

$$\begin{aligned} & i \cdot \omega_L \cdot M_{x,\omega 1} = -\gamma \cdot M_{y,\omega 1} \left[H_z + 4\pi M_{z0} \right] \\ & i \cdot \omega_L \cdot M_{y,\omega 1} = \gamma \cdot H_z \cdot M_{x,\omega 1} \end{aligned} \tag{46}$$

solution of which gives Larmor frequency as

$$\omega_L = \gamma \cdot \sqrt{H_z (H_z + 4\pi M_{z0})} \tag{47}$$

and the ellipticity of precession trajectory for precession at 1st harmonic as

$$\frac{M_{y,\omega 1}}{M_{x,\omega 1}} = -i \cdot R_{yx} = -i \cdot \gamma \frac{H_z}{\omega_L} M_{x,\omega 1} \tag{48}$$

where the ellipticity is calculated as

$$R_{yx} = \sqrt{\frac{H_z}{H_z + 4\pi M_{z0}}} \tag{49}$$

It is the same result as described by Eq. 32

Substitution of Eq. 48 into Eq. 45 gives along-easy-axis component osculating with double Larmor frequency as

$$M_{z,\omega 2} = \frac{4\pi \cdot \gamma}{i \cdot 2\omega_L} \left[-i \cdot \gamma \frac{H_z}{\omega_L} \right] M_{x,\omega 1}^2 = -\frac{2\pi \cdot \gamma^2 H_z}{\omega_L^2} M_{x,\omega 1}^2 \tag{50}$$

Substitution of Eq. 47 into Eq. 50 gives the ratio between components osculating with single and double Larmor frequencies as

$$M_{z,\omega 2} = -i \frac{2\pi}{H_z + 4\pi M_{z0}} M_{x,\omega 1}^2 \tag{51}$$

Next, the magnitude and ellipticity of the 3rd harmonic is calculated. The 2nd Eq. of Eqs 42 and Eq. 48 give

$$M_{y,\omega 3} = \frac{\gamma H_z}{i \cdot 3\omega_L} M_{x,\omega 3} = -i \frac{R_{yx}}{3} M_{x,\omega 3} \tag{52}$$

Eq. 52 the ellipticity of precession trajectory for precession at 3rd harmonic $R_{yx,\omega 3}$ as

$$R_{yx,\omega 3} = i \cdot \frac{M_{y,\omega 3}}{M_{x,\omega 3}} = \frac{R_{yx}}{3} \tag{53}$$

$R_{yx,\omega 3}$ is three times smaller than R_{yx}, meaning that the ellipticity of trajectory for the 3rd harmonic is larger than for the 1st harmonic.

Substitution of Eqs. 48, 51 and 52 into 2nd Eq of Eqs. 39:

$$\begin{aligned} i \cdot 3\omega_L \cdot M_{x,\omega 3} = -\gamma \cdot M_{y,\omega 3} \left[H_z + 4\pi M_{z0} \right] - \\ -\gamma \cdot M_{y,\omega 1} \cdot 4\pi M_{z,\omega 2} \end{aligned} \tag{54}$$

gives

$$\begin{aligned} i \cdot 3\omega_L \cdot M_{x,\omega 3} = -\gamma \left[-i \tfrac{R_{yx}}{3} M_{x,\omega 3} \right] \left[H_z + 4\pi M_{z0} \right] - \\ -\gamma 4\pi \left[-i \cdot R_{yx} \cdot M_{x,\omega 1} \right] \left[-i \tfrac{2\pi}{H_z + 4\pi M_{z0}} M_{x,\omega 1}^2 \right] \end{aligned} \tag{55}$$

Dividing both sides by γ simplifies Eq. 55 to:

$$\begin{aligned} M_{x,\omega 3} \left[i \cdot \tfrac{3\omega_L}{\gamma} - i \tfrac{R_{yx}}{3} \left[H_z + 4\pi M_{z0} \right] \right] = \\ = M_{x,\omega 1}^3 4\pi R_{yx} \tfrac{2\pi}{H_z + 4\pi M_{z0}} \end{aligned} \tag{56}$$

Dividing both sides by R_{yx} and using Eq. 47 simplify Eq. 56 to:

$$\begin{aligned} i \cdot M_{x,\omega 3} \left[3 \tfrac{\sqrt{H_z (H_z + 4\pi M_{z0})}}{R_{yx}} - \tfrac{H_z + 4\pi M_{z0}}{3} \right] = \\ = M_{x,\omega 1}^3 \tfrac{8\pi^2}{H_z + 4\pi M_{z0}} \end{aligned} \tag{57}$$

Substitution of R_{yx} from Eq. 49 gives:

$$i \cdot M_{x,\omega 3} \left[H_z + 4\pi M_{z0} \right] \tfrac{8}{3} = M_{x,\omega 1}^3 \tfrac{8\pi^2}{H_z + 4\pi M_{z0}} \tag{58}$$

So the ratio between components osculating with single and triple Larmor frequencies is calculated as

$$M_{x,\omega 3} = -i \frac{\pi^2}{3} \frac{M_{x,\omega 1}^3}{\left[H_z + 4\pi M_{z0} \right]^2} \tag{59}$$

IV. ACCOUNTING THE SPIN- ORBIT INTERACTION AND REALISTIC DEMAGNETIZATION FIELD

In the previous sections, the Kittel effect was evaluated under the assumption of an ideally smooth, planar interface with no interfacial spin–orbit interaction. This assumption is a substantial oversimplification. Such an ideal interface does not exist in real materials. In practice, surfaces are never perfectly flat, and interfacial spin–orbit interaction is often significant.

A. spin-orbit interaction and imperfection of interface included into Landau-Lifshitz equations

In the “ideal interface” approximation, the demagnetizing field H_{demag} is assumed to be exactly equal in magnitude and opposite in direction to the magnetic field produced by the magnetization M itself (i.e., the field generated by the spins):

$$H_{demag,y} = 4\pi \cdot M_y \tag{60}$$

It is convenient to use unit system, in which the magnetization M and the magnetic field is measured in the same unit. In this case the coefficient 4π is not required in Eq. 60

$$H_{demag,y} = M_y \tag{61}$$

A realistic interface unavoidably has a roughness and other imperfections. It result in a smaller demagnetization field, which is described by a demagnetization coefficient k_{demag}. The demagnetization field becomes

$$H_{demag,y} = k_{demag} \cdot M_y \tag{62}$$

where k_{demag} equals to one for the deal interface and is smaller that one for a realistic interface.

The direction of the demagnetization field is perpendicular to the interface and there is no in-plane component of the demagnetization:

$$H_{demag,x} = H_{demag,z} = 0 \tag{63}$$

Additionally, the electrons experience the magnetic field of spin- orbit interaction. In the presence of the bulk or interracial anisotropy, $\vec{H}_{so}$ can be described using the tensor $\hat{k}_{so}$ as :

$$\vec{H}_{so} = \hat{k}_{so} (\vec{H} + \vec{M} - \vec{H}_{demag}) \tag{64}$$

The y-axis is established as perpendicular to the plane, while the x- and z- axes are set within the plane. In the case of an amorphous nanomagnet, anisotropy in spin-orbit interaction can occur only between the y- and x- (z-) axes. Consequently, the tensor $\hat{k}_{so}$ can be written as follows:

$$\hat{k}_{so} = \begin{pmatrix} k_{so,x} & 0 & 0 \\ 0 & k_{so,y} & 0 \\ 0 & 0 & k_{so,x} \end{pmatrix} \tag{65}$$

The total magnetic field, which an electron experiences, is the sum of an external, internal and demagnetization fields

$$\vec{H}_{total} = \vec{H}_{ext} + \vec{M} - \vec{H}_{demag} + \vec{H}_{so} \tag{66}$$

Substitution Eqs. into Eq. 66 gives

$$\begin{array}{l} H_{total,y} = H_{ext,y} + M_y - k_{demag} M_y + \\ +k_{so,y} \left[H_{ext,y} + M_y - k_{demag} M_y \right] \\ H_{total,x} = H_{ext,x} + M_x + k_{so,x} \left[H_{ext,x} + M_x \right] \\ H_{total,z} = H_{ext,z} + M_z + k_{so,x} \left[H_{ext,z} + M_z \right] \end{array} \tag{67}$$

or

$$\begin{array}{l} H_{total,y} = (1 + k_{so,y}) \left[H_{ext,y} + M_y - k_{demag} M_y \right] \\ H_{total,x} = (1 + k_{so,x}) \left[H_{ext,x} + M_x \right] \\ H_{total,z} = (1 + k_{so,x}) \left[H_{ext,z} + M_z \right] \end{array} \tag{68}$$

To simplify analysis, let us assume that the external magnetic field is applied along the z-axis the total magnetic field is align in-plane and along the z-axis:

$$\vec{H}_{ext} = \begin{pmatrix} 0 \\ 0 \\ H_z \end{pmatrix} \tag{69}$$

Then

$$\begin{array}{l} H_{total,y} = (1 + k_{so,y}) \left[M_y - k_{demag} M_y \right] \\ H_{total,x} = (1 + k_{so,x}) M_x \\ H_{total,z} = (1 + k_{so,x}) \left[H_z + M_z \right] \end{array} \tag{70}$$

The Landau-Lifshitz (LL) equation describes the dynamic of the magnetization precession, induced by total magnetic field H, which includes the internal and external magnetic fields, and the demagnetization field:

$$\frac{\partial \vec{M}}{\partial t} = -\gamma \vec{M} \times \vec{H}_{total} \tag{71}$$

or

$$\begin{array}{l} \frac{\partial M_x}{\partial t} = -\gamma \left[M_y H_{total,z} - M_z H_{total,y} \right] \\ \frac{\partial M_y}{\partial t} = -\gamma \left[M_z H_{total,x} - M_x H_{total,z} \right] \\ \frac{\partial M_z}{\partial t} = -\gamma \left[M_x H_{total,y} - M_y H_{total,x} \right] \end{array} \tag{72}$$

or

$$\begin{array}{l} -\frac{1}{\gamma} \frac{1}{1+k_{so,x}} \frac{\partial M_x}{\partial t} = \\ = \left[M_y (H_{ext,z} + M_z) - M_z H_{total,y} \right] \\ -\frac{1}{\gamma} \frac{1}{1+k_{so,x}} \frac{\partial M_y}{\partial t} = \\ = \left[M_z H_{total,x} - M_x (H_{ext,z} + M_z) \right] \\ -\frac{1}{\gamma} \frac{1}{1+k_{so,x}} \frac{\partial M_z}{\partial t} = \\ = \left[M_x H_{total,y} - M_y H_{total,x} \right] \end{array} \tag{73}$$

$$\begin{pmatrix} \frac{\partial M_x}{\partial t} \\ \frac{\partial M_y}{\partial t} \\ \frac{\partial M_z}{\partial t} \end{pmatrix} = \\ = \frac{-\gamma}{1+k_{so,x}} \begin{pmatrix} M_y (H_z + M_z) - M_z (1 + k_{so})(1 - k_{demag}) M_y \\ M_z M_x - M_x (H_z + M_z) \\ M_x (1 + k_{so})(1 - k_{demag}) M_y - M_y M_x \end{pmatrix} \tag{74}$$

where effective coefficient k_{so} of SO is defined as:

$$1 + k_{so} = \frac{1 + k_{so,y}}{1 + k_{so,x}} \tag{75}$$

Eq. 74 is simplified as

$$\begin{pmatrix} \frac{\partial M_x}{\partial t} \\ \frac{\partial M_y}{\partial t} \\ \frac{\partial M_z}{\partial t} \end{pmatrix} = \\ = \frac{-\gamma}{1+k_{so,x}} \begin{pmatrix} M_y H_z - M_y M_z \left[(1 + k_{so})(1 - k_{demag}) - 1 \right] \\ -M_x H_z \\ M_x M_y \left[(1 + k_{so})(1 - k_{demag}) - 1 \right] \end{pmatrix} \tag{76}$$

It is important to notice that Eq. 76 becomes exactly the same as Eq. 19

$$\begin{array}{l} \frac{\partial M_x}{\partial t} = -\gamma \left[M_y \cdot H_z + M_z \cdot 4\pi M_y \right] \\ \frac{\partial M_y}{\partial t} = -\gamma \left[-M_x \cdot H_z \right] \\ \frac{\partial M_z}{\partial t} = -\gamma \left[-M_x \cdot 4\pi M_y \right] \end{array} \tag{77}$$

when parameters are replaced as

$$\begin{array}{l} \gamma \to \frac{\gamma}{1+k_{so}} \\ 4\pi \to 1 - (1 + k_{so})(1 - k_{demag}) \end{array} \tag{78}$$

In a film with the in-plane equilibrium magnetization, the anisotropy field H^0_{ani} in absence of any external magnetic field is calculated as

$$H^0_{ani} = \frac{2M}{1 + k_{so}} \left[1 - (1 + k_{so})(1 - k_{demag}) \right] \tag{79}$$

Substitution of of Eq. 79 into the 2nd Eq of Eqs. 78 gives

$$\begin{array}{l} \gamma \to \frac{\gamma}{1+k_{so}} \\ 4\pi \to \dfrac{H^0_{ani}(1 + k_{so})}{2 \cdot M} \end{array} \tag{80}$$

B. Calculation of Larmor frequency

Substitution of Eq. 80 into Kittel formula for the Larmor frequency Eq. 26 gives

$$\omega_L = \frac{\gamma}{1 + k_{so}} \cdot \sqrt{H_z (H_z + H^0_{ani} \frac{(1 + k_{so})}{2})} \tag{81}$$

In absence of the interfacial spin- orbit interaction or in the case of isotropic spin-orbit interaction ($k_{so,x} = k_{so,y} = k_{so,z}$), which leads to $k_{so} = 0$, the anisotropy field is calculated as

$$H^0_{ani} = 2M \cdot k_{demag} \tag{82}$$

and the Larmor frequency Eq. 81 becomes

$$\omega_L = \gamma \cdot \sqrt{H_z(H_z + M \cdot k_{demag})} \tag{83}$$

In case of an ideally smooth interface ($k_{demag} = 1$), Eq. 83 is simplified to the classical Kittel formula for the Larmor frequency:

$$\omega_L = \gamma \cdot \sqrt{H_z(H_z + M)} \tag{84}$$

C. ellipticity of precession trajectory

Substitution of Eq. 80 into Eq. 49 gives
the ellipticity of precession trajectory R_{yx} as

$$R_{yx} = \sqrt{\frac{H}{H + H^0_{ani}\frac{(1+k_{so})}{2}}} \tag{85}$$

In absence of the interfacial spin- orbit interaction $k_{so} = 0$, from Eq. 82 the ellipticity is calculated as

$$R_{yx} = \sqrt{\frac{H}{H + M \cdot k_{demag}}} \tag{86}$$

D. 2nd and 3rd harmonics of precession

Substitution of Eq. 80 into Eq. 51 gives the ratio between components osculating with single and double Larmor frequencies as

$$M_{z,\omega 2} = -i\frac{H^0_{ani}(1+k_{so})}{4 \cdot M}\frac{1}{H + \dfrac{H^0_{ani}(1+k_{so})}{2}}M^2_{x,\omega 1} \tag{87}$$

Simplification of Eq. 87 gives

$$M_{z,\omega 2} = -i\frac{M^2_{x,\omega 1}}{2M}\frac{1}{1 + \dfrac{H}{H^0_{ani}}\dfrac{2}{1+k_{so}}} \tag{88}$$

In absence of the interfacial spin- orbit interaction $k_{so} = 0$, from Eq. 82 the ratio between components osculating with single and double Larmor frequencies is calculated as

$$\begin{aligned} M_{z,\omega 2} &= \frac{-i}{2M}\frac{1}{1 + \dfrac{H}{M \cdot k_{demag}}}M^2_{x,\omega 1} = \\ &= -i\frac{0.5}{M + \frac{H}{k_{demag}}}M^2_{x,\omega 1} \end{aligned} \tag{89}$$

Substitution of Eq. 80 into Eq. 59 gives the ratio between components osculating with single and triple Larmor frequencies is calculated as

$$M_{x,\omega 3} = \frac{-i}{3 \cdot 16}\left[\frac{H^0_{ani}(1+k_{so})}{2 \cdot M}\right]^2\frac{M^3_{x,\omega 1}}{\left[H + \dfrac{H^0_{ani}(1+k_{so})}{2}\right]^2} \tag{90}$$

In absence of the interfacial spin- orbit interaction $k_{so} = 0$, from Eq. 90 the ratio between components osculating with single and triple Larmor frequencies is calculated as

$$M_{x,\omega 3} = -i\frac{M^3_{x,\omega 1}}{48 \cdot M^2}\frac{1}{\left[1 + \dfrac{H}{H^0_{ani}}\dfrac{2}{1+k_{so}}\right]^2} \tag{91}$$