Ferromagnetic Resonance (FMR) Excited By A Circularly Polarized Wave With Direction Of Polarization Rotation Matched Direction Of Magnetization Precession

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This is an FMR (Ferromagnetic Resonance) calculation, which describes the magnetization dynamics of a ferromagnetic material under illumination by an electromagnetic wave. The analysis is for the case where the magnetic field component of the wave rotates in the same direction as the natural magnetization precession—representing the most effective polarization for FMR excitation. The main part of this work was completed on January 15, 2025.

I. CONDITIONS, ASSUMPTIONS, AXIS DIRECTIONS

(condition 1):The equilibrium magnetization is perpendicular to the plane (the z- direction). It is the case of a relatively thin ferromagnetic field, when the interfacial anisotropy overcomes the demagnetization field.

(condition 2): Oscillating Magnetic field is circular polarized, and its polarization is rotating in the same direction as the magnetization rotation in the xy-plane, which is perpendicular to the easy axis.

(condition 3): The bias magnetic field H_z and, therefore, the Larmor frequency ω_L do not depend either on precession angle θ or precession phase φ. It means that the variation of precession angle is small and there is no Kittel effect.

(approximation 1): At first, the calculations are done ignoring the precession damping. Later, the precession damping is included as the damping torque.

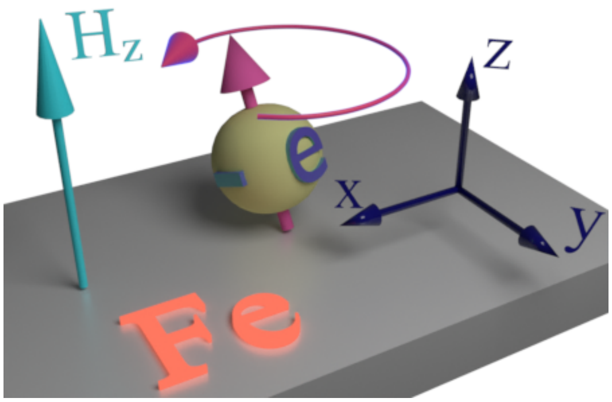

FIG. 1. Precession geometry. The equilibrium magnetization is perpendicular to the film (the z- direction). The external magnetic field H_z is applied along the easy axis (the z- direction). The precession is clockwise direction with respect to the external magnetic field (when to look from back to tip of the arrow)

II. FINAL RESULTS

During the magnetization precession around the z-axis, components of the magnetization are described as:

$$\begin{aligned} m_x(t) &= M \cdot \sin(\theta(t)) \cdot \cos(\omega_L t + \varphi(t)) \\ m_y(t) &= M \cdot \sin(\theta(t)) \cdot \sin(\omega_L t + \varphi(t)) \\ m_z(t) &= M \cdot \cos(\theta(t)) \end{aligned} \tag{1}$$

where m_x,m_y,m_z are component of magnetization M. The easy magnetic axis, and thus the precession axis, is aligned along the z-axis. The precession angle, θ, varies with time. In the case of ferromagnetic resonance (FMR), θ oscillates periodically around a relatively small average precession angle. In the case of parametric reversal, the average precession angle increases continuously until full magnetization reversal occurs. The precession frequency, ω_L, also varies with time. Typically, ω_L is larger for smaller precession angles θ and decreases as θ increases. The precession phase, φ, varies with time. When φ is in phase with the oscillations of the magnetic field of the pumping electromagnetic wave, the precession angle θ increases. When φ is out of phase, the precession angle θ decreases.

The magnetization precession is mathematically described by a solution to the Landau-Lifshitz equation for a system driven by an electromagnetic field of frequency ω. The temporal dynamics are governed by a set of two coupled differential equations. The first equation describes the torque-induced evolution of the precession angle θ, while the second governs the evolution of the precession phase ϕ. These equations, which must be solved simultaneously, are:

$$\begin{aligned} \frac{\partial\theta}{\partial t} &= \Omega_{MW} \cdot \sin[(\omega - \omega_L)t - \varphi] \\ \frac{\partial\varphi}{\partial t} &= -\Omega_{MW} \frac{1}{\tan(\theta)} \cos[(\omega - \omega_L)t - \varphi] \end{aligned} \tag{2}$$

where $\omega_L = \gamma H_z$ is the Larmor frequency, $\Omega_{MW} = \gamma H_{MW}$ is the precession pumping strength and H_{MW} is the magnetic component of the pumping microwave electromagnetic field;

It is important to note that the solution (2) was obtained from Landau- Lifshitz equation without usage of any approximations. Set of differential equations Eqs. 2

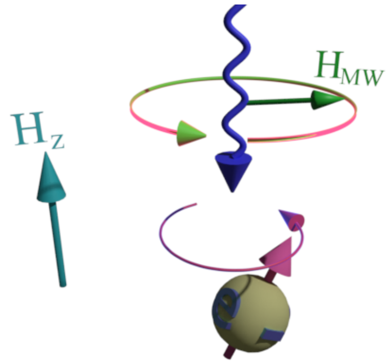

FIG. 2. Geometry of illuminating microwave field. The plane of polarization rotation (the xy-plane) is the same as the plane of the spin precession. The polarization rotation (clockwise direction with respect to the external magnetic field H_z) is the same as direction of the spin precession.

is non-linear and should be solved either numerically or using some approximations.

III. SOLVING LL EQUATIONS. NO APPROXIMATIONS IN USE

Ferromagnetic resonance (FMR) is the process whereby magnetization precession is driven in a ferromagnetic material by the application of an electromagnetic field, most commonly in the radio frequency (RF) band from 2 to 20 GHz. Only the magnetic component of the electromagnetic field interacts with the magnetization.

The magnetization precession occurs around a bias perpendicular magnetic field $H_z = H_{ext} + H_{int}$, where H_{int} is the internal unchanged magnetic field and H_{ext} is the bias perpendicular magnetic field.

The precession pumping is generated by the oscillating magnetic field H_{MW} of the electromagnetic field, which illuminates the ferromagnet.

The Landau-Lifshitz (LL) equation describes the dynamic of the nanomagnet magnetization $\vec{m}$ as:

$$\frac{\partial \vec{m}}{\partial t} = -\gamma \vec{m} \times (\vec{H} + \vec{H}_{MW}) \tag{3}$$

where γ is the electron gyromagnetic ratio, $\vec{H}$ is unchanged bias magnetic field and $\vec{H}_{MW}$ is oscillating magnetic field with the frequency ω (the magnetic component of the pumping microwave field), ω is the frequency of the microwave field, the microwave field is circular- polarized with magnetic field circulary rotating in the same direction as the spin precession (See Fig. 2) :

$$\vec{H} = \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ H_z \end{pmatrix} \tag{4}$$

$$\vec{H}_{MW} = H_{MW} \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \\ 0 \end{pmatrix} \tag{5}$$

$$\vec{m} = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} \tag{6}$$

Then

$$\vec{m} \times \vec{H} = H_z \begin{pmatrix} m_y \\ -m_x \\ 0 \end{pmatrix} \tag{7}$$

and

$$\vec{m} \times \vec{H}_{MW} = H_{MW} \left[\sin(\omega t) \begin{pmatrix} -m_z \\ 0 \\ m_x \end{pmatrix} + \cos(\omega t) \begin{pmatrix} 0 \\ m_z \\ -m_y \end{pmatrix} \right] \tag{8}$$

The scalar form of the Eq. 3 is

$$\begin{aligned} \tfrac{\partial m_x}{\partial t} &= -\omega_L m_y + \Omega_{MW} \cdot m_z \sin(\omega t) \\ \tfrac{\partial m_y}{\partial t} &= \omega_L m_x - \Omega_{MW} \cdot m_z \cos(\omega t) \\ \tfrac{\partial m_z}{\partial t} &= \Omega_{MW} \left[m_y \cos(\omega t) - m_x sin(\omega t) \right] \end{aligned} \tag{9}$$

where the Larmor frequency $\omega_L = \gamma H_z$, which is the precession frequency, and $\Omega_{MW} = \gamma H_{MW}$ is the precession pumping strength of the pumping microwave. It should be noted that ω_L is substantially larger than Ω_{MW}. If ω_L is typically around 10 GHz, Ω_{MW} is much smaller of about 1 MHz and less.

Introduction of new unknowns

$$\begin{aligned} m_+ &= m_x + i \cdot m_y & m_- &= m_x - i \cdot m_y \\ m_x &= \tfrac{m_+ + m_-}{2} & m_y &= \tfrac{m_+ - m_-}{2i} \end{aligned} \tag{10}$$

and addition/ subtraction of the 1st and 2nd equations of Eqs. (9) give

$$\begin{aligned} \tfrac{\partial m_+}{\partial t} &= i\omega_L m_+ + \Omega_{MW} \cdot m_z \left[\sin(\omega t) - i \cdot \cos(\omega t) \right] \\ \tfrac{\partial m_-}{\partial t} &= -i\omega_L m_- + \Omega_{MW} \cdot m_z \left[\sin(\omega t) + i \cdot \cos(\omega t) \right] \\ \tfrac{\partial m_z}{\partial t} &= \Omega_{MW} \left[\tfrac{m_+ - m_-}{2i} \cos(\omega t) - \tfrac{m_+ + m_-}{2} \sin(\omega t) \right] \end{aligned} \tag{11}$$

or

$$\begin{aligned} \tfrac{\partial m_+}{\partial t} &= i\omega_L m_+ - i \cdot \Omega_{MW} \cdot m_z e^{i \cdot \omega t} \\ \tfrac{\partial m_-}{\partial t} &= -i\omega_L m_- + i \cdot \Omega_{MW} \cdot m_z e^{-i \cdot \omega t} \\ \tfrac{\partial m_z}{\partial t} &= \tfrac{\Omega_{MW}}{2i} \left[m_+ e^{-i \cdot \omega t} - m_- e^{+i \cdot \omega t} \right] \end{aligned} \tag{12}$$

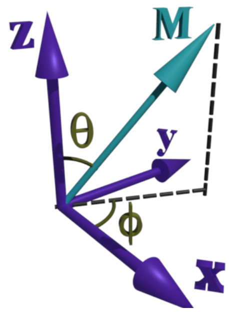

FIG. 3. Precession geometry. θ is the precession angle of magnetization M with respect to the easy axis (the z- axis). The ϕ is the precession phase with respect to the x- axis.

The solution of Eq. 12 can be found in form of the magnetization precession, which can be described as

$$\begin{aligned} m_x(t) &= M \cdot \sin(\theta(t)) \cdot \cos(\phi(t)) \\ m_y(t) &= M \cdot \sin(\theta(t)) \cdot \sin(\phi(t)) \\ m_z(t) &= M \cdot \cos(\theta(t)) \end{aligned} \tag{13}$$

where $\theta(t)$ and $\phi(t)$ are new time- dependent unknowns and $M = \sqrt{m_x^2 + m_y^2 + m_z^2}$ is the magnetization, which is time-independent: $\frac{\partial M}{\partial t} = 0$ (See Fig. 3).

Summing and substituting 1st and 2nd Eqs. of Eqs. 13 gives:

$$\begin{aligned} m_+(t) &= M \cdot \sin(\theta(t)) \cdot e^{i\phi(t)} \\ m_-(t) &= M \cdot \sin(\theta(t)) \cdot e^{-i\phi(t)} \\ m_z(t) &= M \cdot \cos(\theta(t)) \end{aligned} \tag{14}$$

Differentiating Eqs. 14 gives

$$\begin{aligned} \frac{\partial m_+}{\partial t} &= M \cdot e^{i\phi} \left(\cos(\theta) \frac{\partial \theta}{\partial t} + i \frac{\partial \phi}{\partial t} \sin(\theta) \right) \\ \frac{\partial m_-}{\partial t} &= M \cdot e^{-i\phi} \left(\cos(\theta) \frac{\partial \theta}{\partial t} - i \frac{\partial \phi}{\partial t} \sin(\theta) \right) \\ \frac{\partial m_z}{\partial t} &= M \cdot \left(-\sin(\theta) \frac{\partial \theta}{\partial t} \right) \end{aligned} \tag{15}$$

Substitution of Eqs. 15, 13 into Eq. 12 and dividing over M gives

$$\begin{aligned} &\left[\cos(\theta) \frac{\partial \theta}{\partial t} + i \frac{\partial \phi}{\partial t} \sin(\theta) \right] e^{i\phi} = \\ &= i\omega_L \sin(\theta) e^{i\phi} - i \cdot \Omega_{MW} \cdot \cos(\theta) e^{+i\cdot\omega t} \\ &\left[\cos(\theta) \frac{\partial \theta}{\partial t} - i \frac{\partial \phi}{\partial t} \sin(\theta) \right] e^{-i\phi} = \\ &= -i\omega_L \sin(\theta) e^{-i\phi} + i \cdot \Omega_{MW} \cdot \cos(\theta) e^{-i\cdot\omega t} \\ &-\sin(\theta) \frac{\partial \theta}{\partial t} = \frac{\Omega_{MW}}{2i} sin(\theta) \left[e^{+i\cdot\phi} e^{-i\cdot\omega t} - e^{-i\cdot\phi} e^{+i\cdot\omega t} \right] \end{aligned} \tag{16}$$

dividing the both sides of 1st equation over $\cos(\theta)e^{i\phi}$ and both sides of the 2nd equation over $\cos(\theta)e^{-i\phi}$ and 3rd equation over $\sin(\theta)$ give

$$\begin{aligned} &\frac{\partial \theta}{\partial t} + i \frac{\partial \phi}{\partial t} \tan(\theta) = i\omega_L \tan(\theta) - i \cdot \Omega_{MW} \cdot e^{i\cdot(\omega t - \phi)} \\ &\frac{\partial \theta}{\partial t} - i \frac{\partial \phi}{\partial t} \tan(\theta) = -i\omega_L \tan(\theta) + i \cdot \Omega_{MW} \cdot e^{i\cdot(-\omega t + \phi)} \\ &\frac{\partial \theta}{\partial t} = \Omega_{MW} \cdot \sin(\omega t - \phi) \end{aligned} \tag{17}$$

It is important to note that only two of three equations of Eqs.(17) are independent. It is because summing the 1st and 2nd equations gives the 3rd equation. The 3rd equation and subtraction of the 1st and 2nd equations gives the following system of two differential equations:

$$\begin{aligned} &i \frac{\partial \phi}{\partial t} \tan(\theta) = \\ &= i\omega_L \tan(\theta) - i \cdot 0.5 \cdot \Omega_{MW} \left[e^{+i\cdot(\omega t - \phi)} + e^{-i\cdot(\omega t - \phi)} \right] \\ &\frac{\partial \theta}{\partial t} = \Omega_{MW} \cdot \sin(\omega t - \phi) \end{aligned} \tag{18}$$

Dividing the 1st equation over $i \cdot \tan(\theta)$ gives the solution of LL equations as

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \omega_L - \Omega_{MW} \frac{1}{\tan(\theta)} \cos(\omega t - \phi) \\ \frac{\partial \theta}{\partial t} &= \Omega_{MW} \cdot \sin(\omega t - \phi) \end{aligned} \tag{19}$$

It is important to note that the solution (19) was obtained from LL equations (3) without usage of any approximations

In the absence of the oscillating magnetic field $H_{MW} = 0$ (absence of the microwave pump), which leads to and $\Omega_{MW} = 0$. Eq.19 becomes

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \omega_L \\ \frac{\partial \theta}{\partial t} &= 0 \end{aligned} \tag{20}$$

which has a solution

$$\begin{aligned} \phi &= \omega_L t + \phi_0 \\ \theta &= \theta_0 \end{aligned} \tag{21}$$

It describes the magnetization precession at a constant angle θ_0 at the Larmor frequency ω_L

Introduction of new independents as

$$\begin{aligned} \phi &= \omega_L t + \varphi(t) \\ \theta &= \theta_0 + \theta_1(t) \end{aligned} \tag{22}$$

simplifies Eq. 19 as

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= -\Omega_{MW} \frac{1}{\tan(\theta_0 + \theta_1(t))} \cos[(\omega - \omega_L)t - \varphi(t)] \\ \frac{\partial \theta_1}{\partial t} &= \Omega_{MW} \cdot \sin[(\omega - \omega_L)t - \varphi(t)] \end{aligned} \tag{23}$$

This is the final solution. The 2nd Eq. describes the torque inserted by the microwave on spin precession. The torque forces the precession angle θ either to increase or to decrease. The 1st equation describes the modulation of the precession phase.

Set of differential equations Eqs. 23 is non-linear and should be solved either numerically or using some approximations.

It is important to note that the solution (23) was obtained from LL equations (3) without usage of any approximations

A. Symmetry of solution

Since the solution in Eq. 23 was derived directly from the Landau–Lifshitz equation (Eq. 3) without any approximations, the set of two differential equations in Eq. 23 is fully equivalent to the original set of three differential equations in Eq. 3.

Symmetry 1: The solution is periodic with period $\frac{2\pi}{\omega-\omega_L}$

Since replacing the time variable as $t \to t+\frac{2\pi}{\omega-\omega_L}$ leaves Eq. 23 unchanged, any solution of Eq. 23 is therefore periodic with period $\frac{2\pi}{\omega-\omega_L}$

Symmetry 2: The torque in the first half-period is equal in magnitude but opposite in sign to the torque in the second half-period. Consequently, the net torque over one full period is zero.

Since the following replacement

$$\begin{aligned} & t \to -t \\ & \varphi \to -\varphi \\ & \frac{\partial \theta}{\partial t} \to -\frac{\partial \theta}{\partial t} \\ & \theta \to \theta \end{aligned} \tag{24}$$

leaves Eq. 23 unchanged, The torque $\frac{\partial \theta}{\partial t}$ in the first half-period is equal in magnitude but opposite in sign to the torque in the second half-period. Consequently, the precession angle returns to its initial value after each period.

B. Average and maximum torque and deviation of precession angle.

Beating period:

$$T_{beating} = \frac{2\pi}{\omega-\omega_L} = \frac{1}{f-f_L} \tag{25}$$

where f is frequency of electromagnetic field and f_L is the Larmor frequency.

The beating period is the time interval over which the torque reverses its polarity and then returns to its original value. During the first half of the period, energy is transferred from the electromagnetic wave to the spin precession, while during the second half, the energy is transferred back from the spin precession to the electromagnetic wave.

It should be noted that the Larmor frequency decreases as the precession angle increases. For example, when the precession angle is 90 degrees, the energies of the spin-up and spin-down states are identical, and the Larmor frequency vanishes. Consequently, the beating period shortens as the precession angle grows. The minimum beating period is reached at a precession angle of, where it becomes:

$$T_{beating,min} = \frac{1}{f} \tag{26}$$

Maximum torque: From the second equation of Eq. 23 the maximum torque within one beating period is given as:

$$\left[\frac{\partial \theta}{\partial t}\right]_{maximum} = \Omega_{MW} \tag{27}$$

Average torque: In the absence of precession or phase damping, the net torque averaged over one full beating period is **zero**. The positive torque during the first half of the period is exactly canceled by the negative torque during the second half.

When there is precession damping, the average torque is positive and equal to the average damping torque.

Average positive torque: in 1st half of beating period

$$\left[\frac{\partial \theta}{\partial t}\right]_{pos,aver} = 2\frac{\Omega_{MW}}{T_{beating}}\int_0^{T_{beating}/2} \sin[(\omega-\omega_L)t-\varphi(t)]\cdot dt \tag{28}$$

where the initial phase is chosen as $\varphi(t=0)=0$

Notably, the average positive torque increases with a reduction in the beating period. This reduction occurs when the electromagnetic wave's frequency approaches the Larmor frequency.

Maximum precession angle:

The precession reaches its maximum precession angle θ_{max} at the end of the first half of the beating period, precisely when the torque changes from positive to negative.

$$\theta_{max} = \Omega_{MW}\int_0^{T_{beating}} \sin[(\omega-\omega_L)t-\varphi(t)]\cdot dt \tag{29}$$

assuming that the minimum precession angle is zero and $\varphi(t=0)=0$.

Magnetization reversal:

A precession angle greater than 90° causes the equilibrium magnetization to invert. As a result, when the driving electromagnetic field is turned off, the magnetization relaxes not back to its initial alignment, but to the opposite direction. The minimum intensity of electromagnetic field required for magnetization reversal can be found from Ω_{MW} , which itself can be calculated from the following equation::

$$\frac{\pi}{2} = \Omega_{MW}\int_0^{T_{beating}} \sin[(\omega-\omega_L)t-\varphi(t)]\cdot dt \tag{30}$$

Since both $T_{beating}$ and ω_L depend on precession angle, Eq. 30 should be calculated numerically.

Energy of the electromagnetic wave absorbed to sustain magnetization precession:

In the case of the stable magnetization precession, the energy loss due to the precession damping is compensated by energy absorbed from the pumping electromagnetic wave.

The magnetization precession occurs around a bias perpendicular magnetic field $H_z = H_{ext} + H_{int}$, where H_{int} is the internal unchanged magnetic field and H_{ext} is the bias perpendicular magnetic field. In this case the magnetic energy is calculated as:

$$E = \vec{H}_z \cdot \vec{M} = (H_{ext} + H_{int}) \cdot M \cdot cos(\theta) \tag{31}$$

The change of energy in time is calculated as

$$\frac{\partial E}{\partial t} = -H_z \cdot M \cdot sin(\theta)\frac{\partial \theta}{\partial t} + M \cdot cos(\theta)\frac{H_{int}}{\partial t} \tag{32}$$

Assuming that the internal field is independent of the precession angle, Eq. 32 is simplified as

$$\frac{\partial E}{\partial t} = -H_z \cdot M \cdot sin(\theta)\frac{\partial \theta}{\partial t} \tag{33}$$

Substitution of the torque of the pumping electromagnetic field the 2nd Eq of Eqs. 23 into Eq. 33 gives the average energy per the beating period, which is asorbed from the electromagnetic field, as

$$\begin{aligned} &E_{absorbed} = \\ &H_z \cdot \Omega_{MW} \int_0^{T_{beating}} sin(\theta(t)) \cdot \sin[(\omega - \omega_L)t - \varphi(t)] \cdot dt \end{aligned} \tag{34}$$

IV. CALCULATION OF TORQUE USING APPROXIMATIONS

The first approximation is relatively simple and straightforward. It assumes that a certain magnetization precession already exists, and that variations in the precession angle occur only as small oscillations around this average precession angle. In this approximation, the change in the precession angle is considered much smaller than the average precession angle itself.

$$\theta_0 >> \theta_1 \tag{35}$$

Then, the dynamic equation (23) are simplified to

$$\begin{aligned} \tfrac{\partial \varphi}{\partial t} &= -\Omega_{MW} \tfrac{1}{\tan(\theta_0)} \cos[(\omega - \omega_L)t - \varphi] \\ \tfrac{\partial \theta_1}{\partial t} &= \Omega_{MW} \cdot \sin[(\omega - \omega_L)t - \varphi] \end{aligned} \tag{36}$$

This approximation is not strict and well fits to the conditions of realistic precession

A. The zero approximation : Ignoring the change of the precession phase

This approximation assumes that the phase of magnetization precession remains unchanged under the influence of pumping. Due to a phase mismatch between the oscillations of the electromagnetic wave and the magnetization precession, the torque alternates periodically between positive and negative values. As a result, the pumping and damping effects largely cancel each other out, leading to no net pumping. Consequently, the precession angle oscillates around a constant average value.

Ignoring the phase change in the 2nd equation of Eqs. 36 and choosing a constant phase as $\varphi = 0$ gives the torque as

$$\frac{\partial \theta}{\partial t} = \Omega_{MW} \cdot \sin[(\omega - \omega_L)t] \tag{37}$$

The torque is oscillating from maximum negative value of $-\Omega_{MW}$ to maximum positive value $+\Omega_{MW}$ within time period $\frac{2\pi}{\omega - \omega_L}$. Torque averaged over the period equals to zero. Therefore, in average the magnetization does not experience any torque. Within the period, the torque reverses its polarity.

Average positive torque can be calculated as

$$\left[\frac{\partial \theta}{\partial t}\right]_{pos,aver} = 2 \cdot \Omega_{MW} \tag{38}$$

The solution of Eq. (37), gives the oscillating precession angle around θ_0 as

$$\theta = \theta_0 - \frac{\Omega_{MW}}{\omega - \omega_L} \cos[(\omega - \omega_L)t] \tag{39}$$

Often the precession damping makes the minimum precession angle to be zero, then the variation of the precession angle is described as

$$\theta = \frac{\Omega_{MW}}{\omega - \omega_L}(1 - \cos[(\omega - \omega_L)t]) \tag{40}$$

The precession angle θ is oscillating from zero to maximum positive value $\frac{2\Omega_{MW}}{\omega - \omega_L}$. The amplitude of the change of the precession angle becomes larger when the frequency of electromagnetic wave ω approaches the Larmor frequency ω_L.

The condition of the magnetization reversal Eq. 30 becomes:

$$\frac{2\Omega_{MW}}{\omega - \omega_L} > \frac{\pi}{2} \tag{41}$$

While this simplified approximation allows for infinitely large variations in the precession angle amplitude, when ω is close to ω_L,—an unrealistic outcome. In a

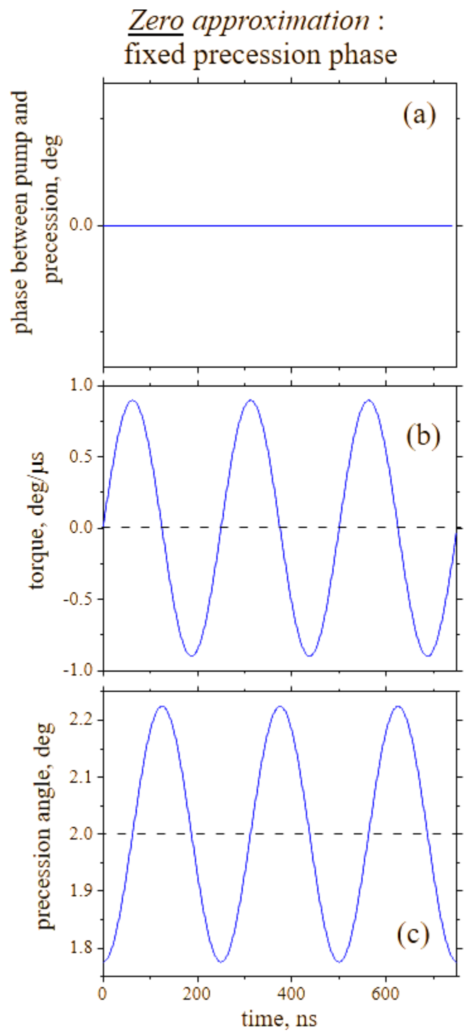

FIG. 4. Approximation 0. (a) Phase difference between pump and precession. (b) precession pumping torque. It has equal positives and negative parts, resulting in no average torque (c) Precession angle, which oscillates around an average angle of 2 degrees. The pumping frequency is 4 MHz above the Larmor frequency $\omega - \omega_L = 4MHz$.

physically accurate scenario, however, the amplitude of the precession angle variation is constrained and remains relatively small (see below).

Substitution of Eqs. 37 and 40 into Eq. 33 gives change of precession energy as:

$$\begin{aligned} \frac{\partial E}{\partial t} &= -H_z \cdot M \cdot \Omega_{MW} \cdot \sin[(\omega - \omega_L)t] \cdot \\ &\cdot sin\left[\frac{\Omega_{MW}}{\omega-\omega_L}(1 - \cos[(\omega - \omega_L)t])\right] \end{aligned} \tag{42}$$

Figure 4 shows the calculated magnetization precession under the zero-order approximation, for the case where the pumping frequency is 4 MHz below the Larmor frequency. The amplitude of the oscillating magnetic component of the electromagnetic wave is 5.6 mGauss.

In this approximation, the phase of the precession is assumed constant over time. Due to the frequency mismatch between the pumping signal and the precession, the pumping torque oscillates symmetrically between positive and negative values, resulting in zero net average torque. As a result, the precession angle oscillates around the initial value of 2 degrees without growing. From 0 to 125 ns, when the torque is positive, the precession angle increases from 1.77 to 2.22 degrees. In the second half of the period when the negative the precession angle decreases back to 1.77 degrees.

B. The first approximation : There is a modulation of precession phase, but such modulation is simplified

This approximation is softer than the zero approximation of previous chapter. It allows the variation of phase φ, but this variation is smaller than the phase variation $(\omega - \omega_L)t$ due to the frequency beating. The approximation is described as:

$$|(\omega - \omega_L)t| >> |\varphi(t)| \tag{43}$$

The approximation is valid either when the pumping frequency ω is relatively far from the Larmor frequency ω_L or when the calculated time interval t is short or when the pumping strength Ω_{MW} is small (See below Eq.(47)).

The approximation (43) simplifies the 1st equation of Eqs. 36

$$\frac{\partial \varphi}{\partial t} = -\Omega_{MW} \frac{1}{\tan(\theta_0)} \cos[(\omega - \omega_L)t - \varphi(t)] \tag{44}$$

to

$$\frac{\partial \varphi}{\partial t} = -\Omega_{MW} \frac{1}{\tan(\theta_0)} \cos[(\omega - \omega_L)t - \varphi_0] \tag{45}$$

where φ_0 is a time independent constant.

Solution of the equation (50), gives the oscillating phase as

$$\varphi(t) = -\frac{\Omega_{MW}}{\omega - \omega_L} \frac{1}{\tan(\theta_0)} \sin[(\omega - \omega_L)t - \varphi_0] \tag{46}$$

Substitution of Eq.(46) into (43) gives more concrete limitation of validity of the 1st approximation as

$$|(\omega - \omega_L)t| >> \left| \frac{\Omega_{MW}}{\omega - \omega_L} \frac{1}{\tan(\theta_0)} \sin[(\omega - \omega_L)t - \varphi_0] \right| \tag{47}$$

Substitution of Eq.(46) into the 2nd equation of Eqs. 36 gives the torque of the 1st approximation as

$$\begin{aligned} \frac{\partial \theta_1}{\partial t} &= \Omega_{MW} \cdot \\ &\cdot \sin\left\{(\omega - \omega_L)t + \frac{\Omega_{MW}}{\omega-\omega_L} \frac{1}{\tan(\theta_0)} \sin[(\omega - \omega_L)t]\right\} \end{aligned} \tag{48}$$

Several important symmetry properties of the torque Eq. (48) of the 1st approximation should be noted:

(property 1): The torque is still oscillating from maximum negative value of $-\Omega_{MW}$ to maximum positive value $+\Omega_{MW}$ within time period $\frac{2\pi}{\omega-\omega_L}$

It is because the transformation $t \rightarrow t+\frac{2\pi}{\omega-\omega_L}$ does not change the Eq. 48 for the torque

(property 2): Torque averaged over the period still equals to zero.

It is because the torque is still asymmetric for time reversal $t \rightarrow -t$

$$\frac{\partial\theta_1}{\partial t}(t) = -\frac{\partial\theta_1}{\partial t}(-t) \tag{49}$$

Over the period the positive torque exactly equals to the negative torque and in the average there is no torque

(property 3): The precession phase φ is oscillating with the same period $\frac{2\pi}{\omega-\omega_L}$ (See Eq. 46). The largest variation of the phase is $\pm\frac{\Omega_{MW}}{\omega-\omega_L}\frac{1}{\tan(\theta_0)}$. The variation becomes larger, when microwave intensity is larger, when microwave frequency ω is closer to the Larmor frequency ω_L and when the precession angle θ becomes smaller.

C. The second approximation; more complex dynamic for the precession phase φ

.

This approximation is softer than the 1st approximation of previous chapter. It accounts the variation of phase φ in the right of the 1st differential equation of Eqs.(36):

$$\frac{\partial\varphi}{\partial t} = -\Omega_{MW}\frac{1}{\tan(\theta_0)}\cos[(\omega-\omega_L)t - \varphi_0 - \varphi(t)] \tag{50}$$

where φ_0 is a time independent constant and $\varphi(t=0) = 0$

The Eq.(50) can be simplified using the following trigonometric relation:

$$\begin{aligned}&\cos[(\omega-\omega_L)t-\varphi(t)] = \\ &= \cos[(\omega-\omega_L)t]\cos[\varphi(t)] + \sin[(\omega-\omega_L)t]\sin[\varphi(t)]\end{aligned} \tag{51}$$

The approximation is described as:

Note: This approximation is valid at least within shout time interval.

It is valid over the whole period of the torque oscillation when

$$\frac{\Omega_{MW}}{\omega-\omega_L}\frac{1}{\tan(\theta_0)} << 1 \tag{52}$$

Therefore it is valid over whole period when when microwave intensity is larger, when microwave frequency ω is closer to the Larmor frequency ω_L and when the precession angle θ becomes smaller.

and when

$$\sin[\varphi(t)] << 1 \tag{53}$$

the 1st equation of Eqs. (36)

$$\frac{\partial\varphi}{\partial t} = -\Omega_{MW}\frac{1}{\tan(\theta_0)}\cos[(\omega-\omega_L)t - \varphi(t)] \tag{54}$$

is simplified to

$$\frac{\partial\varphi}{\partial t} = -\Omega_{MW}\frac{1}{\tan(\theta_0)}\cos[(\omega-\omega_L)t]\cos[\varphi(t)] \tag{55}$$

The differential equation 55 can be solved as

$$\frac{\partial\varphi}{\cos(\varphi)} = -\Omega_{MW}\frac{\partial t}{\tan(\theta_0)}\cos[(\omega-\omega_L)t] \tag{56}$$

Integration gives

$$\begin{aligned}&\ln\left[\frac{\ln[\cos(\varphi/2)-\sin(\varphi/2)]}{\cos(\varphi/2)-\sin(\varphi/2)}\right] = \\ &= -\frac{\Omega_{MW}}{\omega-\omega_L}\frac{1}{\tan(\theta_0)}\sin[(\omega-\omega_L)t]\end{aligned} \tag{57}$$

or

$$\begin{aligned}&\frac{\ln[\cos(\varphi/2)-\sin(\varphi/2)]}{\cos(\varphi/2)-\sin(\varphi/2)} \\ &= e^{-\frac{\Omega_{MW}}{\omega-\omega_L}\frac{1}{\tan(\theta_0)}\sin[(\omega-\omega_L)t]}\end{aligned} \tag{58}$$

V. INTRODUCTION OF PRECESSION DAMPING

In absence of precession damping

VI. CALCULATION OF THE MAGNETIC FIELD COMPONENT OF AN ELECTROMAGNETIC WAVE

The relationship between the electric and magnetic fields in a plane wave in vacuum is:

$$H = \frac{E}{Z_0} \tag{59}$$

where $Z_0 \approxeq 377\,Ohm$ is is the impedance of free space.

For example, when an EM wave has an electric field amplitude $E = 1\,V/mm$, $H \approx 2.65\,A/m \approx\; 0.033\,Gauss$

For another example, when an EM wave has an electric field amplitude $E = 1V/nm$ (e.g. 1V is applied to a MgO tunnel junction), $H \approx 2.65\,MA/m \approx\; 33.3\,kGauss$

In the case of conventional impedance of 50 Ohm, the magnetic field of the microwave is calculated as

$$H(A/m) = \frac{E(V/m)}{50\,Ohm} \tag{60}$$

The unit conversion is

$$B(Gauss) = \mu_0 \cdot H = 4\pi \cdot 10^{-3} \cdot H(A/m) \tag{61}$$

or

$$B(Gauss) = 0.0125 \cdot H(A/m) \tag{62}$$